Mastering Mathematics

13.04.16
Mastery and the National Curriculum

• become **fluent** in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils have **conceptual understanding** and are able to recall and apply their knowledge rapidly and accurately to problems

• **reason** mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language

• can **solve problems** by applying their mathematics to a variety of **routine and non-routine problems with increasing sophistication**, including breaking down problems into a series of simpler steps and persevering in seeking solutions.

Children must master the curriculum for their year group, so that they have firm foundations to build on the following year.
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Table 1.1 TIMSS 2011 performance groups: mathematics at ages 9-10
Source: Exhibit 1.3 international mathematics report
NCETM
The approach to teaching maths in high performing jurisdictions such as Shanghai and Singapore is based on a few fundamentals.

- Problem solving curriculum
- Emphasis on the development of intellectual competence such as the ability to visualise and heuristics (practical approach to problem solving)
- Emphasis on conceptual understanding
- Systematic development of skills and concepts
- Emphasis on the C - P - A approach

The wise men of mathematics

Concrete – Pictorial – Abstract.

Jerome Bruner. American. 1915 –


Conceptual and procedural understanding.

Using what we already know

**David Ausubel:** American - 1918 - 2008

Origins of thinking

**Jean Piaget:** Swiss – 1896 – 1980

Multiple intelligence

**Martin Gardner:** American: 1914 - 2010

Social theory

**Lev Vygotsky:** Russian – 1896 - 1934

Practical apparatus

**Georges Cuisenaire:** Belgium: 1891 - 1975
What is mastery?

If you drive a car, imagine the process you went through...

- The very first drive, lacking the knowledge of what to do to get moving
- The practice, gaining confidence that you are able to drive
- The driving test, fairly competent but maybe not fully confident
- A few years on, it’s automatic, you don’t have to think about how to change gears or use the brake
- Later still, you could teach someone else how to drive or drive in any situation
What is mastery?

Mastery in mathematics is similar.

It involves:

• Deep and sustainable learning
• Ability to build on something already mastered
• Ability to reason about a concept and make connections to other concepts
• Procedural and conceptual fluency (can’t solve problems without these)
• The understanding of how and why it all works

Mastery is a continuum... mastery at a particular point of time that is sufficient mastery for that stage of learning and then built on at a later stage
What is mastery?

Teaching for Mastery...

• The focus is on the development of deep structural knowledge and the ability to make connections. Making connections in mathematics deepens knowledge of concepts and procedures, ensures what is learnt is sustained over time, and cuts down the time required to assimilate and master later concepts and techniques. NCETM 2014
Mastery curriculum

• Practice makes perfect, perfect practice makes permanent - spending longer on key concepts
  – Variation
  – Intelligent practice, e.g., practice within different contexts, opportunities to develop fluency
  – Extended practice which goes deeper... and deeper
  – Practice to spot relationships and make connections
  – Practice to deepen conceptual understanding
  – Meeting the needs of all children – intelligent differentiation

• Practice and consolidation within different contexts, e.g. time, money, length.

• There is a focus on the development of deep structural knowledge and the ability to make connections.
Relational understanding

• Show something physically with a model, e.g. $8 + 6$ with objects, straws, Numicon etc.
• Draw something appropriate
• Explain orally
• Explain in writing with diagrams
• Think of ways to challenge themselves, e.g. how else can I work out $8 + 6$
The expectation is that the majority of pupils will move through the programmes of study at broadly the same pace. However, decisions about when to progress should always be based on the security of pupils’ understanding and their readiness to progress to the next stage. Pupils who grasp concepts rapidly should be challenged through being offered rich and sophisticated problems before any acceleration through new content. Those who are not sufficiently fluent with earlier material should consolidate their understanding, including through additional practice, before moving on.
Language and Communication

Teaching precise mathematical **content** vocabulary, e.g.

- Augend + addend = sum
- Minuend – subtrahend = difference
- Multiplicand x multiplier = product
- Dividend ÷ divisor = quotient
- Division bracket
- Vinculum, denominator, numerator

Using precise mathematical **process** vocabulary

- Compare, estimate, regroup, partition, rearrange, exchange
What can we do to make these equal?
Mastery requires knowledge

There are three forms of knowledge:

– Factual: I know...
– Procedural: I know how...
– Conceptual: I know why...

The children don’t need to just know if the answer is correct, they need to know why.
Resources to represent quantity and the visualisation of number facts
Resources to represent quantity and the visualisation of number facts
Tens Frames
The children have been counting in ones, fives and tens since Year R. So the multiplication facts for 1 and 10 could be learned at the end of Year R and those for 5 should be learned in Year 1.

Square numbers

When children learn about halves in fractions, it makes sense for them to learn their multiplication facts for 2. When they learn about quarters it makes sense for them to learn facts for 4. Also 3s when learning about thirds.

Double 3s for 6s
Double 6s for 12s
Double 4s for 8s

Commutativity
Tables
Tables
Where mastery begins......

Counting:

- Stable order principle
- One to one principle
- Cardinal principle
- Order irrelevance principle
- Abstraction principle

Along with this:
Knowing all about numbers to 10
Ability to subitise
Number recognition

Key performance indicators for Year R

Gelman and Gallistel
Understanding the equals sign

the same as

equivalent

equal

balance

Not the answer to a calculation!
Understanding place value

• Positional
• Multiplicative
• Additive
• Base 10

(Ross 1989)
Do these represent an understanding of place value?

\[ \underline{61 \div 1256} \]

\[ 54 + 29 = \]

JULIA ANGHILERI, MEINDERT BEISHUIZEN and KEES VAN PUTTEN (2001)
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Well known mental calculation strategies

• Partition and recombine
• Doubles and near doubles
• Use number pairs to 10 and 100
• Adding near multiples of ten and adjusting
• Using patterns of similar calculations
• Using known number facts
• Bridging though ten, hundred, tenth
Bridging 10

7 + 6 by making 10 and something – a mental calculation strategy from NNS days rarely used!

\[ 7 + 6 = 13 \]

\[ 10 + 3 = 13 \]
Bridging 10

16 + 8 by making 10 and something

10 + 10 + 4 = 24
Bridging 10

13 - 6 by making 10 and something

13 - 6 = 13

10 - 3 = 7
Mastery requires the use of manipulatives

What’s the same?

What’s different?
Mastery requires the use of manipulatives
Mastery requires the use of manipulatives

Arrays help to show the commutativity of multiplication and its inverse relationship with division

- 5 lots of 4
  - $4 \times 5$
  - $4 + 4 + 4 + 4 + 4 = 20$

- 4 lots of 5
  - $5 \times 4$
  - $5 + 5 + 5 + 5 = 20$
Mastery requires the use of manipulatives

$18 \times 3$

What’s different about these models?
What is the same?
Mastery requires the use of manipulatives

20 ÷ 4 = 5
20 – 4 – 4 – 4 – 4

20 ÷ 5 = 4
20 – 5 – 5 – 5 – 5
Mastery requires the use of manipulatives

$20 \div 4 = 5$
$20 - 4 - 4 - 4 - 4 - 4 - 4$

$20 \div 5 = 4$
$20 - 5 - 5 - 5 - 5 - 5$
Why do children find telling the time difficult?

They don’t need to tell it!

It uses base 60 not 10

We confuse them!

Tips to help...
Give them a reason to tell the time
Mean the time you say!
Focus on minutes past not to:
• that will come later
• digital time
• number lines
Follow the date trail below to find out when Sam’s birthday is.
F means go forward. B means go back.
Find the quickest way to move around the calendar, e.g. instead of counting 21 days move along 3 weeks.
   Start on 2\textsuperscript{nd} February.
   F 5 days
   B 24 hours
   F 48 hours
   F15 days
   B 24 hours
   F 21 days
   B 24 hours
   F 35 days
   B 7 days
   F 24 hours.
You have now landed on Sam’s birthday!
40 minutes past 2 o’clock

2:40

20 minutes to 3 o’clock

Link to 5x table

Link to digital

Work on minutes past

Investigate number of minutes to next hour
What is a fraction?

\[ \frac{4}{16} \]

\[ \frac{2}{8} \]

\[ \frac{1}{4} \]

\[ 1 \div 4 \]

25%

0.25
What is a fraction?

There are several meanings to consider:

A relationship between two quantities: part/whole model
A fraction as a division: quotient model
A fraction as an operator
A fraction as a number (cardinal and ordinal)
A fraction as a proportion
A fraction as a probability
Most importantly, fractions express relationships

Take the idea of a quarter. I can use this to think about:

- The quarter of my cake I’m trying not to eat
- How I’m a quarter of the way through the assignments I’m marking
- That I like salad dressing to be one tablespoon of vinegar to three tablespoons of oil
- So if I want to make a large jar, a quarter of it needs to be vinegar
- How the shrub I planted is about a quarter of the height to which it will grow, according its label.

The thing to note is that a ‘quarter’ in these examples expresses a relationship. You don’t need to know how many assignments, how much dressing I need or what type of shrub I planted, but you can make sense of a ‘quarter’ each time. Fractions are tools for thinking about and describing relationships; they are not things or objects. Except, unfortunately, in primary school.

Excerpt of article by Mike Askew
Fractions in the National Curriculum

The programmes of study identify key aspects of the fractions curriculum.

- Understand and recognise what a fraction is;
- Connect different types;
- Read, write and use the language of fractions;
- Equivalence;
- Compare and order;
- Calculate, for example, find $\frac{2}{3}$;
- Connect to division.
National Curriculum requirements for KS1

Year 1:
• Recognise, find and name a half as one of two equal parts of an object, shape or quantity
• Recognise, find and name a quarter as one of four equal parts of an object, shape or quantity

Year 2:
• Recognise, find, name and write $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{4}$ and $\frac{3}{4}$ of a length, shape, set of objects or quantity
• Write simple fractions e.g. $\frac{1}{2}$ of 6 = 3 and recognise the equivalence of $\frac{2}{4}$ and $\frac{1}{2}$
The word fraction actually comes from the Latin "fractio" which means to break.
Part/whole model

The first thing children need to know...

Whole...part relationship

The UK is the whole....
England is the part
Part/whole model

England is the whole...
Yorkshire is the part
Part/whole model

What is the whole/part relationship here?
Part/whole model: making connections to division

1. Draw the vinculum to show you are breaking something into parts. In the same way as you would write a division symbol.

2. Write the denominator (always call it this) to show how many parts there are.

3. Write the numerator (always call it this) to show how many parts are needed.
Fractions of what?

When dealing with fractions…

… we need to help the children make connections, so, for example, looking at half in lots of different ways.

• Shapes
• Discrete objects
• Continuous measures
Bad practice!

Worksheets like these help to develop misconceptions.
Good practice!
Addition and subtraction

Year 2 reasoning about fractions video
Exploring fractions
Mastery requires the children to be able to make generalisations

This bar model can then help the children solve, for example, missing number problems:

\[35 + ? = 68, \ ? - 42 = 13, \ 50 - ? = 20, \ ? + 15 = 58\]
Freddy scored 59 points on the computer game. Samir scored 34 points more. How many points did Samir score?

Jenny had a collection of shells. She gave her friend 23 of them. She was left with 46. How many shells did she have before giving some to her friend?

Sam went for a run. He ran 12km to the shop and 7km back. Then he stopped to talk to a friend. How much further did he need to run to get home.
More about the bar model...

- It is a mathematical representation of a word problem

- It is a representation that reveals the structure of a word problem

- A way of ‘acting’ out a problem

- It is not a calculating tool
Number problems always:

• contain **part** and **whole** relationships.

• contain **knowns** and **unknowns**

Peter has 4 books.
Harry has five times as many books as Peter.
How many more books has Harry?
Peter has 4 books.
Harry has five times as many books as Peter. How many more books has Harry?
Part-Whole Model for addition and subtraction

Sofia has 7 sweets and Finley has 5 sweets.

How many sweets do they have altogether?

2

7
5
Part-Whole Model for addition and subtraction

47 boys and 35 girls attended a concert. Find the total number of children at the concert.
Joshua has 12 sweets. Sophie has 10 sweets. How many more sweets does Joshua have than Sophie?
Comparison Model

The difference between two numbers is 48. If the larger number is 3 times the smaller number, find the sum of the two numbers.
Josie had 3 times as many sweets as Abi. Josie gave Abi some of her sweets. They now each have 20. How many sweets did Josie have before sharing them with Abi?

Sam had 5 times as many marbles as Tom. If Sam gives 10 marbles to Tom, the two friends will have exactly the same amount. How many marbles do they have altogether?
Part-whole Model for multiplication and division

The farmer has 24 animals.
There are three times as many sheep as cows.
How many sheep and how many cows?

Another farmer has 42 animals
There are twice as many ducks as cows and three times as many sheep as cows.
How many sheep, cows and ducks?
Part-whole Model for fractions
A gardener plants tulip bulbs in a flower bed.

She plants 3 red bulbs for every 4 white bulbs.

She plants 60 red bulbs.

How many **white** bulbs does she plant?
2014 SATs Level 6 Paper 2

4 Two numbers are in the ratio 4 : 5

One of the numbers is 60

There are two possible values for the other number.

What are the two possible values?
Change/Transform Model

A shop keeper sold 1/3 of his balloons in the afternoon and 2/5 of the remainder in the evening.

If he had 150 balloons left, find the number of balloons he had at first.
Hannah baked some biscuits. She packed 2/3 of them into a tin and gave 1/5 of the remainder to her friend. She had 40 biscuits left. How many biscuits did she bake?

Mr Yap had a length of rope. He used 1/4 of it to tie some boxes together. He then used 5/9 of the remainder to make a skipping rope for his daughter. 120cm of rope were left. What was the length of rope used to tie the boxes together?

Michelle prepared a mixture of apple, carrot and celery juices. 1/3 of the mixture was apple juice and 2/5 of the remainder was celery juice. 315 ml of the mixture was celery juice. What volume of the mixture was carrot juice?
Iqbal and Sofia have £680 altogether. If Iqbal spends \( \frac{2}{5} \) of her money and Sofia spends £80, then they will have an equal amount of money left. How much money did Sofia have at first?
Sophie made some cakes for the school fair. She sold $\frac{3}{5}$ of them in the morning and $\frac{1}{4}$ of what was left in the afternoon. If she sold 200 more cakes in the morning, how many cakes did she make?
Manipulatives and visual representations ‘open the door’ to conceptual understanding and should be used with all children. This will then lead to the procedural fluency and the mastery that the new National Curriculum requires.

Teaching rules alone does not give children the conceptual understanding that they need.